Pulse propagation – guide to solution

1) Consider the spectral representation of the pulse $f(\omega) = \int_{-\infty}^{+\infty} f(t) e^{i\omega t} dt$
What is the effect of a time translation $t \to t \pm \tau$ on $f(\omega)$?

We note $f_\tau(t)$ the envelope of a pulsed that is delayed by τ compared to $f(t)$:
$$f_\tau(t+\tau) = f(t) \Leftrightarrow f_\tau(t) = f(t-\tau)$$
(since it is true $\forall t$)

The Fourier transform of the delayed pulse is :
$$f_\tau(\omega) = \int_{-\infty}^{+\infty} f_\tau(t) e^{i\omega t} dt = \int f(t-\tau) e^{i\omega t} dt = \int f(t') e^{i\omega(t'+\tau)} dt'$$
$$= f(\omega) e^{i\omega\tau}$$

In other words :
$$\boxed{\mathcal{F}\left[f(t-\tau)\right](\omega) = \mathcal{F}\left[f(t)\right](\omega) \cdot e^{i\omega\tau}}$$
or
$$\boxed{f(t-\tau) \longleftrightarrow f(\omega) e^{i\omega\tau}}$$

We see that a time translation adds a phase linear in ω in Fourier domain. Importantly, the slope is proportional to delay.

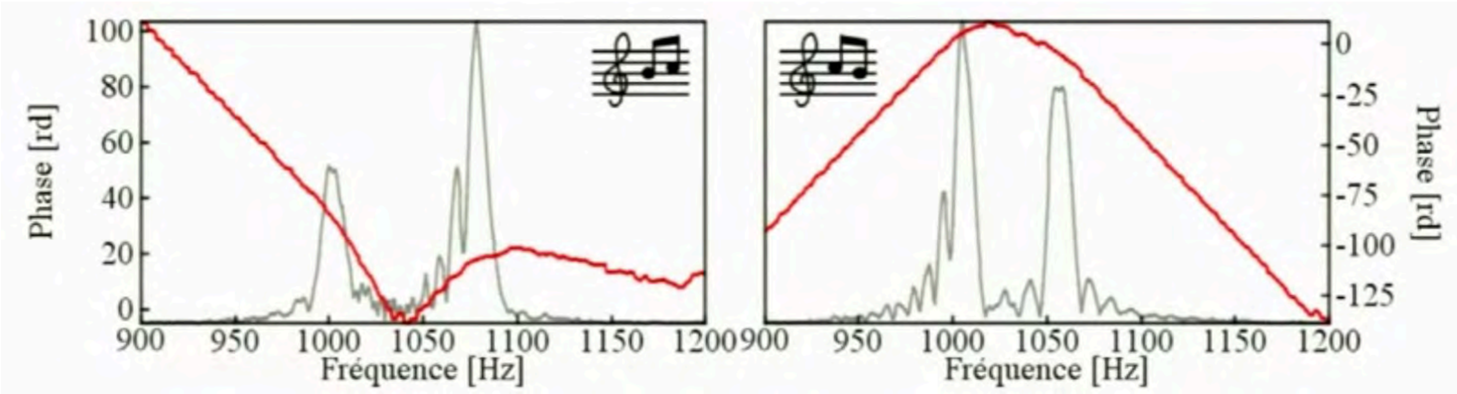
2) The spectrum is dominated by two sharp peaks around
$\omega_B \approx 2\pi \times 990$ Hz and $\omega_C \approx 2\pi \times 1050$ Hz

We can approximate the full spectrum corresponding to the sequence B - C or C - B as : $f(\omega) = |f_B(\omega)| e^{i\omega\tau_B} + |f_C(\omega)| e^{i\omega\tau_C}$

where f_B and f_C are centered on ω_B, ω_C resp.

- In the top graph, the slope of the spectral phase is higher around ω_B than ω_C $\rightarrow$ B comes after C ($\tau_B > \tau_C$)
- The opposite is true for the lower graph : C after B.

The result is summarized on the next page.



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3) $$f(\omega) = |f(\omega)|\, e^{i\varphi(\omega)} \Rightarrow f'(\omega) = \left(\frac{d|f(\omega)|}{d\omega} + i\varphi'(\omega)\,|f(\omega)|\right) e^{i\varphi(\omega)}$$

4) $$\langle t \rangle \overset{\text{def.}}{=} \int_{-\infty}^{+\infty} t\,|f(t)|^2\,dt = -i\int_{-\infty}^{+\infty} f^*(t)\, it\, f(t)\,dt \quad \left\{\begin{array}{l} 1 = -i \times i \\ |f|^2 = f^* f \end{array}\right.$$

$$= -i\int_{-\infty}^{+\infty} f^*(\omega)\, f'(\omega)\, \frac{d\omega}{2\pi} \qquad (\text{Parseval} + f'(\omega) \leftrightarrow it\, f(t))$$

$$= -i\int_{-\infty}^{+\infty} |f(\omega)|\, \frac{d|f(\omega)|}{d\omega}\, \frac{d\omega}{2\pi} + \int_{-\infty}^{+\infty} \varphi'(\omega)\, |f(\omega)|^2 \frac{d\omega}{2\pi} \qquad (3))$$

$$\overset{\text{integr.}}{=} -i\left[\frac{1}{2}|f(\omega)|^2\right]_{-\infty}^{+\infty} + \langle \varphi'(\omega) \rangle^{\text{def.}}$$

But the function f vanishes at $\pm\infty$, therefore we showed:

$$\langle t \rangle = \langle \varphi'(\omega) \rangle = \langle \tau_g \rangle \quad \text{where } \tau_g(\omega) = \frac{\partial \varphi}{\partial \omega} = \varphi'(\omega)$$

In other words, the barycenter of the pulse in the time domain equals the barycenter of the group delay τ_g in the frequency domain, where τ_g is the slope of the spectral phase.

5) We consider a pulse $\boxed{E(z,\omega) = \mathcal{E}(z,\omega)\, e^{-i\omega t} + c.c.}$

with $\mathcal{E}(0,\omega) = E_0\, f(0,\omega)$ at $z = 0$

6) At $z > 0$ in a medium of dispersion $k(\omega)$ a plane wave will evolve according to :

$$E(z,\omega) = E_0\, f(0,\omega)\, e^{ik(\omega)z - i\omega t} + c.c.$$

So that
$$\mathcal{E}(z,\omega) = E_0\, f(0,\omega)\, e^{ik(\omega)z}$$
$$= E_0\, |f(0,\omega)|\, \exp i\left(\varphi(0,\omega) + k(\omega)z\right)$$

i.e. the spectral phase evolves as : $\boxed{\varphi(z,\omega) = \varphi(0,\omega) + k(\omega)z}$

7) We just have to take the mean values (barycenters) after differentiating with respect to ω :

$$\langle t\rangle_z = \langle \varphi'(z,\omega)\rangle = \langle \varphi'(0,\omega)\rangle + \left\langle \frac{\partial k(\omega)}{\partial\omega}\right\rangle z$$

If we note $\boxed{\frac{1}{v_g} = \left\langle \frac{\partial k(\omega)}{\partial \omega}\right\rangle}$ we have indeed : $z = v_g\left(\langle t\rangle_z - \langle t\rangle_{z=0}\right)$

as expected for the pulse velocity

8) only if $\underline{n(\omega) = n_0}$ do we have

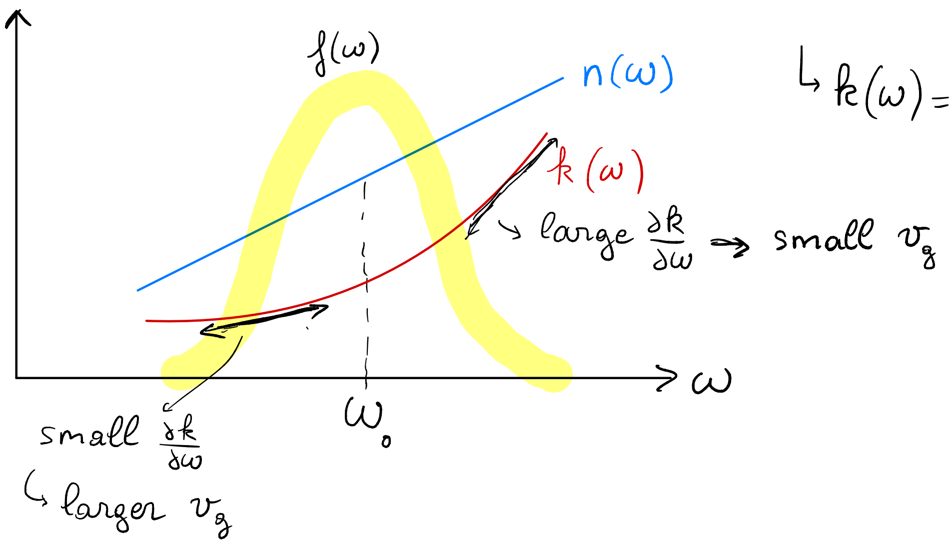
$$\frac{1}{v_g} = \left\langle \frac{n_0}{c}\right\rangle = \frac{n_0}{c} = \text{constant} \rightarrow v_g = \frac{c}{n_0} = \text{phase velocity}$$

A light pulse propagates <u>unchanged</u> , <u>at the phase velocity</u>

9) We assume that locally $n(\omega) = n(\omega_0) + \left.\frac{dn}{d\omega}\right|_{\omega_0} \times (\omega - \omega_0)$

$= n_0 + n_0' (\omega - \omega_0)$ with $n_0' > 0$

$\hookrightarrow k(\omega) = n_0 \frac{\omega}{c} + \underbrace{n_0' \frac{\omega(\omega - \omega_0)}{c}}_{\ni \text{ quadratic term}}$



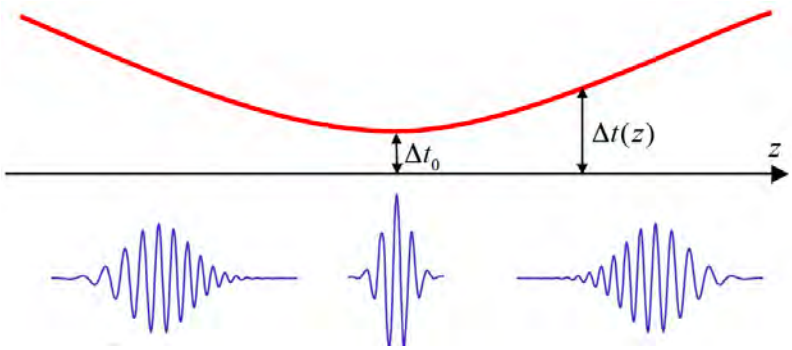
We see that the low frequency side of the pulse travels faster than the high frequency side.

→ A Fourier transform limited pulse will become chirped from red to blue (in time), it will broaden in time.

→ Reciprocally, an incoming pulse chirped from blue to red would "focus" in time to a Fourier transform limited pulse.

↳ useful method to obtain short pulses after going through thick optics ("precompensation")

The plot below illustrates both cases above. The red curve is a measure of pulse duration "$\Delta t(z)$". Notice similarity with diffraction.



Taylor expansion : $k(\omega) = k(\omega_0) + \left.\frac{\partial k}{\partial \omega}\right|_{\omega_0}(\omega-\omega_0) + \frac{1}{2}\left.\frac{\partial^2 k}{\partial \omega^2}\right|_{\omega_0}(\omega-\omega_0)^2$

$$\hookrightarrow \frac{\partial k}{\partial \omega} = \left.\frac{\partial k}{\partial \omega}\right|_{\omega_0} + 2\left.\frac{\partial^2 k}{\partial \omega^2}\right|_{\omega_0}(\omega-\omega_0) = k_0' + k_0''(\omega-\omega_0)$$

Then $\boxed{\frac{1}{v_g} = \left\langle \frac{\partial k}{\partial \omega} \right\rangle = k_0'}$ because $\langle k_0''(\omega-\omega_0)\rangle = k_0''\langle \omega-\omega_0\rangle$

$$= k_0'' \times (\omega_0-\omega_0)$$
$$= 0$$

From the previous discussion, the pulse duration depends on k_0''

In fact it is possible to show that :

$$\Delta t(z) = \sqrt{\Delta t^2(0) + \Delta k'^2 z^2} \quad \text{and} \quad \Delta k'^2 = \langle (k'(\omega)-k_0')^2 \rangle$$
$$= \langle (\omega-\omega_0)^2 k_0''^2 \rangle$$
$$= \Delta\omega^2 k_0''^2$$

so that : $\boxed{\Delta t(z) = \sqrt{\Delta t^2(0) + k_0''^2 . \Delta\omega^2 . z^2}}$

In the limit where $|k_0'' \Delta\omega . z| \gg \Delta t(0)$ then $\Delta t(z) \simeq k_0'' . \Delta\omega . z$

$\hookrightarrow$ It can happen easily for very short pulses.

- For a Fourier transform limited pulse $\Delta\omega . \Delta t(0) = 1/2 \Rightarrow \Delta t(0) = \frac{1}{2\Delta\omega}$

Then the distance z_0 after which the pulse has been broadened by a factor $\sqrt{2}$ is : $z_0 = \frac{\Delta t(0)}{k_0'' \Delta\omega} = \frac{2\Delta t^2(0)}{k_0''}$

$\hookrightarrow$ We see that this distance increases quadratically with pulse duration. In practice, for pulse duration > 100 fs, time broadening is not an issue for typical glass thicknesses $<$ few cm.